Quantum entanglement measure based on wedge product

Hoshang Heydari

Institute of Quantum Science, Nihon University,

1-8 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan

Abstract

We construct an entanglement measure that coincides with the generalized concurrence for a general pure bipartite state based on wedge product. Moreover, we construct an entanglement measure for pure multi-qubit states, which are entanglement monotone. Furthermore, we generalize our result on a general pure multipartite state.

1 Introduction

Quantum entanglement is one of the most interesting properties of quantum mechanics. It has become an essential resource for quantum information (including quantum communication and quantum computing) developed in recent years, with some potential applications such as quantum cryptography [1, 2]and quantum teleportation [3]. Quantification of a multipartite state entanglement [4, 5] is quite difficult and is directly linked to algebra, geometry, and functional analysis. The definition of separability and entanglement of a multipartite state was introduced in[6], following the definition of bipartite states, given by Werner [7]. One of the widely used measures of entanglement of a pair of qubits is entanglement of formation and the concurrence, that gives an analytic formula for the entanglement of formation [8, 9, 10]. In recent years, there have been some proposals to generalize this measure on general pure bipartite states [11, 12, 13, 14] and on multipartite states [15, 16, 17, 18, 19]. There also have been many works on entanglement measures which rely on a wedge product, e.g., through the definition of hyper-determinants. For example, F. Verstraete *et al.* [20] have considered a single copy of a pure four-partite state of qubits and investigated its behavior under stochastic local quantum operation and classical communication(SLOCC)[21], which gave a classification of all different classes of pure states of four qubits. They have also shown that there exist nine families of states corresponding to nine different ways of entangling four qubits. Note that, all homogeneous positive functions of pure states that are invariant under SLOCC operations are called entanglement monotones. F. Verstraete *et al.* [22] have also presented a general mathematical framework to describe local equivalence classes of multipartite quantum states under the action of local unitary and local filtering operations. Their analysis has lead to the introduction of entanglement measures for the multipartite states, and the optimal local filtering operations maximizing these entanglement monotones were obtained. E. Briand, [23] *et.* al have obtained a complete and minimal set of

170 generators for the algebra of $SL(2, \mathbf{C})^{\times 4}$-covariants of a binary quadrilinear form. Interpreted in terms of a four qubit system, this describes in particular the algebraic varieties formed by the orbits of local filtering operations in its projective Hilbert space. E. Briand, [24] *et.* al have also studied the invariant theory of trilinear forms over a three-dimensional complex vector space, and apply it to investigate the behavior of pure entangled three-partite qutrit states and their normal forms SLOCC operations. They described the orbit space of the SLOCC group $SL(3, \mathbf{C})^{\times 3}$ both in its affine and projective versions in terms of a very symmetric normal form parameterized by three complex numbers. They have also shown that the structure of the sets of equivalent normal forms is related to the geometry of certain regular complex polytopes. A. Miyake and M. Wadati [25] have explored quantum search from the geometric viewpoint of a complex projective space. They have shown that the optimal quantum search can be geometrically identified with the shortest path along the geodesic joining a target state, an element of the computational basis, and such an initial state as overlaps equally, up to phases, with all the elements of the computational basis. They have also calculated the entanglement through the algorithm for any number of qubits n as the minimum Fubini-Study distance to the submanifold formed by separable states in Segre embedding, and find that entanglement is used almost maximally for large n. Recently, Péter Lévay [26] have constructed a class of multi-qubit entanglement monotones which was based on construction of C. Emary [27]. His construction is based on bipartite partitions of the Hilbert space and the invariants are expressed in terms of the Plücker coordinates of the Grassmannian. We have also constructed entanglement monotones for multi-qubit states based on Plücker coordinate equations of Grassmann variety, which are central notion in geometric invariant theory [28]. However, we do have different approaches and construction to solve the problem of quantifying multipartite states compare to those of Lévay and Emary. In this paper, we will construct a measure of entanglement by using the algebraic definition of wedge product without touching the geometrical structure of algebraic projective variety defining this measure of entanglement. In particular, in section 3 we will derive a measure of entanglement that coincides with the generalized concurrence for a general pure bipartite state, based on an algebraic point of view, using wedge product, a useful tool from multi-linear algebra, which is mostly used in algebraic and differential geometry and topology, in relation with differential forms. In section 4, we will construct entanglement monotones for a general pure multi-qubit state. Moreover, we generalize our construction on a general pure multipartite state in section 5. To make this note self-contained in section 2, we will give a preview introduction to multilinear algebra. Let us denote a general, multipartite quantum system with m subsystems by $\mathcal{Q} = \mathcal{Q}_m(N_1, N_2, \ldots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of a state $|\Psi\rangle = \sum_{k_1=1}^{N_1}\cdots\sum_{k_m=1}^{N_m}\alpha_{k_1,\ldots,k_m}|k_1,\ldots,k_m\rangle$ and, let $\rho_{\mathcal{Q}} = \sum_{n=1}^{\mathrm{N}} p_n|\Psi_n\rangle\langle\Psi_n|$, for all $0 \leq p_n \leq 1$ and $\sum_{n=1}^{\mathrm{N}} p_n = 1$, denote a density operator acting on the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1}\otimes\mathcal{H}_{\mathcal{Q}_2}\otimes\cdots\otimes\mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is given by $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper, i.e., we denote a mixed pair of qubits by $\mathcal{Q}_2(2,2)$. The density operator $\rho_{\mathcal{Q}}$ is said to be fully separable, which we will denote by $\rho_{\mathcal{Q}}^{sep}$, with respect to the Hilbert space decomposition, if it can be written as $\rho_{\mathcal{Q}}^{sep} = \sum_{n=1}^{\mathrm{N}} p_n \bigotimes_{j=1}^{m}\rho_{\mathcal{Q}_j}^{n}$, $\sum_{n=1}^{\mathrm{N}} p_n = 1$, for some positive integer N, where

p_n are positive real numbers and $\rho^n_{\mathcal{Q}_j}$ denotes a density operator on the Hilbert space $\mathcal{H}_{\mathcal{Q}_j}$. If $\rho^p_{\mathcal{Q}}$ represents a pure state, then the quantum system is fully separable if $\rho^p_{\mathcal{Q}}$ can be written as $\rho^{sep}_{\mathcal{Q}} = \bigotimes_{j=1}^m \rho_{\mathcal{Q}_j}$, where $\rho_{\mathcal{Q}_j}$ is a density operator on $\mathcal{H}_{\mathcal{Q}_j}$. If a state is not separable, then it is called an entangled state. Some of the generic entangled states are called Bell states and EPR states.

2 Multilinear algebra

In this section, we review the definitions and properties of multilinear algebra and exterior algebra. Multilinear algebra extends the methods of linear algebra, which builds on the concept of tensor. The exterior algebra $\bigwedge(V)$ or Grassmann algebra of a given vector space V is a certain unital associative algebra, which contains V as a subspace and its multiplication, known as the wedge product or the exterior product written as $\wedge$. The wedge product is associative and bilinear. Now, let us consider the complex vector spaces $\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m$ to be vector spaces, where $\dim(\mathrm{V}_j) = N_j,\ \forall j = 1, 2, \ldots, m$. Then, we define a tensor of type (m, n) on $\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m$ as follows

$$\begin{aligned}\mathcal{T}^m_n(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m) &= \mathcal{L}(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m; \mathrm{V}^*_1, \mathrm{V}^*_2, \ldots, \mathrm{V}^*_n) \\ &= \mathrm{V}_1 \otimes \mathrm{V}_2 \otimes \cdots \otimes \mathrm{V}_m \otimes \mathrm{V}^*_1 \otimes \mathrm{V}^*_2 \otimes \cdots \otimes \mathrm{V}^*_n,\end{aligned} \tag{1}$$

where $\mathrm{V}^*_j = \mathcal{L}(\mathrm{V}_j; \mathbf{C})\ \forall j = 1, 2, \ldots, m$ is the space of linear applications $\mathrm{V}_j \longrightarrow \mathbf{C}$ and is called dual of V_j. For any basis e_i of V_j and e^j the dual basis of e_i defined by $e^j(e_i) = \delta^j_i$, we have the following linear representation

$$T = T^{i_1, i_2, \ldots, i_m}_{j_1, j_2, \ldots, j_n} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_n}. \tag{2}$$

For example, we have $\mathcal{T}^1_0(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m) = \mathrm{V}_1$ and $\mathcal{T}^0_1(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m) = \mathrm{V}^*_1$. Let S_m be a group of permutations $(1, 2, \ldots, m)$. Then, we call the tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_m \in \mathcal{T}^m_0(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m)$ symmetric if

$$v_1 \otimes v_2 \otimes \cdots \otimes v_m = v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(m)}, \tag{3}$$

for all $\pi \in S_m$. The space of symmetric tensor is denoted by $\mathcal{S}^m_0(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m)$. Moreover, we call the tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_m \in \mathcal{T}^m_0(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m)$ skew-symmetric if

$$v_1 \otimes v_2 \otimes \cdots \otimes v_m = \epsilon(\pi) v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(m)}, \tag{4}$$

for all $\pi \in S_m$, where $\epsilon(\pi)$ is the signature of permutation π. The space of skew-symmetric tensor is denoted by $\Lambda^m_0(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m)$. Furthermore, we have the following mapping

$$\begin{array}{rcl}\mathrm{Alt}^m : \mathcal{T}^m_0(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m) & \longrightarrow & \Lambda^m_0(\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_m) \\ v_1 \otimes v_2 \otimes \cdots \otimes v_m & \longmapsto & v_1 \wedge v_2 \wedge \cdots \wedge v_m\end{array}, \tag{5}$$

where the alternating map is defined by $\mathrm{Alt}^m(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_1 \wedge v_2 \wedge \cdots \wedge v_m = \frac{1}{m!} \sum_{\pi \in S_m} \epsilon(\pi) v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(m)}$. For example, for $m = 2$ we have

$$\begin{array}{rcl}\mathrm{Alt}^2 : \mathcal{T}^2_0(\mathrm{V}_1, \mathrm{V}_2) & \longrightarrow & \Lambda^2_0(\mathrm{V}_1, \mathrm{V}_2) \\ v_1 \otimes v_2 & \longmapsto & \mathrm{Alt}^2(v_1 \otimes v_2) = v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1\end{array}. \tag{6}$$

The essential property of wedge product is that it is alternating on V, that is $v \wedge v = 0$ for all vector $v \in V$ and $v_1 \wedge v_2 \wedge \cdots \wedge v_n = 0$, whenever $v_1, v_2, \ldots, v_n \in V$ are linearly dependent.

3 Measure of entanglement for general bipartite state

In this section, we will directly construct a measure of entanglement for a general pure bipartite state $\mathcal{Q}_2^p(N_1, N_2)$, based on the wedge product. So let $\Lambda_{\mu,\nu} = v_\mu \wedge v_\nu$, $v_\mu = (\alpha_{\mu,1}, \alpha_{\mu,2}, \ldots, \alpha_{\mu,N_2})$, $v_\nu = (\alpha_{\nu,1}, \alpha_{\nu,2}, \ldots, \alpha_{\nu,N_2})$ and $\overline{\Lambda}_{\mu,\nu}$ denote the complex conjugate of $\Lambda_{\mu,\nu}$. Then, a measure of entanglement for a quantum system $\mathcal{Q}_2^p(N_1, N_2)$ is given by

$$\mathcal{E}(\mathcal{Q}_2^p(N_1, N_2)) = \left(\mathcal{N}_2 \sum_{\nu > \mu = 1}^{N_1} \Lambda_{\mu,\nu} \overline{\Lambda}_{\mu,\nu}\right)^{\frac{1}{2}}. \quad (7)$$

Moreover, if we write the coefficients α_{i_1,i_2}, for all $1 \leq i_1 \leq N_1$ and $1 \leq i_2 \leq N_2$ in form of a $N_1 \times N_2$ matrix as below

$$\mathrm{M} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,N_2} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,N_2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N_1,1} & \alpha_{N_1,2} & \cdots & \alpha_{N_1,N_2} \end{pmatrix}, \quad (8)$$

then $\Lambda_{\mu,\nu}(\mathrm{M}) = v_\mu(\mathrm{M}) \wedge v_\nu(\mathrm{M})$, where, the vectors $v_\mu(\mathrm{M})$ and $v_\nu(\mathrm{M})$ refer to different rows of the matrix M. As an example let us look at the quantum system $\mathcal{Q}_2^p(2,2)$ representing a pair of qubits. Then, the expression for a measure of entanglement for such state using the above equation (7) is given by

$$\begin{aligned} \mathcal{E}(\mathcal{Q}_2^p(2,2)) &= \left(\mathcal{N}_2 \Lambda_{1,2}(\mathrm{M}) \overline{\Lambda}_{1,2}(\mathrm{M})\right)^{\frac{1}{2}} = \left(2\mathcal{N} |\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}|^2\right)^{\frac{1}{2}} \quad (9) \\ &= 2|\alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}|, \end{aligned}$$

where $\Lambda_{1,2}(\mathrm{M})$ is given by

$$\begin{aligned} \Lambda_{1,2}(\mathrm{M}) &= v_1 \wedge v_2 = (\alpha_{1,1}, \alpha_{1,2}) \otimes (\alpha_{2,1}, \alpha_{2,2}) - (\alpha_{2,1}, \alpha_{2,2}) \otimes (\alpha_{1,1}, \alpha_{1,2}) \\ &= (0, \alpha_{1,1}\alpha_{2,2} - \alpha_{2,1}\alpha_{1,2}, \alpha_{1,2}\alpha_{2,1} - \alpha_{2,2}\alpha_{1,1}, 0) \quad (10) \end{aligned}$$

for $\mathcal{N}_2 = 2$. The measure of entanglement for the general bipartite state defined in equation (7) coincides with the generalized concurrence

$$\mathcal{C}(\mathcal{Q}_2^p(N_1, N_2)) = \left(4\mathcal{N}_2 \sum_{l_1 > k_1 = 1}^{N_1} \sum_{l_2 > k_2 = 1}^{N_2} |\alpha_{k_1,k_2}\alpha_{l_1,l_2} - \alpha_{k_1,l_2}\alpha_{l_1,k_2}|^2\right)^{1/2} \quad (11)$$

defined in [11, 12, 13, 14], and in particular equation (9), that gives the concurrence of a pair of qubits, first time defined in [9, 10].

4 Entanglement measure for multi-qubit states

In this section, we will construct a measure of entanglement for multi-qubit states, based on exterior product. Let the operator $\Lambda_{1,2}(\mathrm{M}_j) = \upsilon_1(\mathrm{M}_j) \wedge \upsilon_2(\mathrm{M}_j)$ be the wedge product between row number one and two of a given matrix M_j, which is constructed by coefficient of a quantum system $\mathcal{Q}_m^p(2,2,\ldots,2)$, e.g., we define

$$\begin{aligned}
\mathrm{M}_1 &= \left(\begin{array}{cccc} \alpha_{1,1,\ldots,1} & \alpha_{1,1,\ldots,2} & \cdots & \alpha_{1,2,\ldots,2} \\ \alpha_{2,1,\ldots,1} & \alpha_{2,1,\ldots,2} & \cdots & \alpha_{2,2,\ldots,2} \end{array}\right), \\
\mathrm{M}_2 &= \left(\begin{array}{cccc} \alpha_{1,1,\ldots,1} & \alpha_{1,1,\ldots,2} & \cdots & \alpha_{2,1,\ldots,2} \\ \alpha_{1,2,\ldots,1} & \alpha_{1,2,\ldots,2} & \cdots & \alpha_{2,2,\ldots,2} \end{array}\right), \\
&\vdots \\
\mathrm{M}_m &= \left(\begin{array}{cccc} \alpha_{1,1,\ldots,1} & \alpha_{1,1,\ldots,1} & \cdots & \alpha_{2,2,\ldots,1} \\ \alpha_{1,1,\ldots,2} & \alpha_{1,1,\ldots,2} & \cdots & \alpha_{2,2,\ldots,2} \end{array}\right),
\end{aligned} \tag{12}$$

where M_j for $1 \leq j \leq m$ are 2-by-2^{m-1} matrices and, e.g., the matrix M_2 is constructed by permutation of indices of the matrix M_1. Then, we can define a measure of entanglement for multi-qubit states by

$$\begin{aligned}
\mathcal{E}(\mathcal{Q}_m^p(2,2\ldots,2)) &= \left(\mathcal{N}_m \sum_{j=1}^{m} \Lambda_{1,2}(\mathrm{M}_j)\overline{\Lambda}_{1,2}(\mathrm{M}_j)\right)^{1/2} \qquad (13) \\
&= \left(\sum_{j=1}^{m}[\upsilon_1(\mathrm{M}_j) \wedge \upsilon_2(\mathrm{M}_j)]\overline{[\upsilon_1(\mathrm{M}_j) \wedge \upsilon_2(\mathrm{M}_j)]}\right)^{1/2},
\end{aligned}$$

where $\mathcal{N}_m$ is a normalization constant. This measure of entanglement is entanglement monotones. However, an algebraic proof of this statement seems difficult, but this can be seen from geometrical structure called Grassmann variety, which is constructed by the Plücker coordinate equations [27, 26, 28]. In this geometrical construction one define a measure of entanglement in terms of Plücker coordinate equations, which is invariant under action of SLOCC by construction. As an example, let us consider the quantum system $\mathcal{Q}_3^p(2,2,2)$. For such three-qubit states, if for instance the subsystem $\mathcal{Q}_1$ is unentangled with subsystems $\mathcal{Q}_2\mathcal{Q}_3$, then we have

$$\mathrm{M}_1 = \left(\begin{array}{cccc} \alpha_{1,1,1} & \alpha_{1,1,2} & \alpha_{1,2,1} & \alpha_{1,2,2} \\ \alpha_{2,1,1} & \alpha_{2,1,2} & \alpha_{2,2,1} & \alpha_{2,2,2} \end{array}\right), \ \mathrm{M}_2 = \left(\begin{array}{cccc} \alpha_{1,1,1} & \alpha_{1,1,2} & \alpha_{2,1,1} & \alpha_{2,1,2} \\ \alpha_{1,2,1} & \alpha_{1,2,2} & \alpha_{2,2,1} & \alpha_{2,2,2} \end{array}\right), \tag{14}$$

and $\mathrm{M}_3 = \left(\begin{array}{cccc} \alpha_{1,1,1} & \alpha_{2,1,1} & \alpha_{1,2,1} & \alpha_{2,2,1} \\ \alpha_{1,1,2} & \alpha_{2,1,2} & \alpha_{1,2,2} & \alpha_{2,2,2} \end{array}\right)$. To illustrate this construction let us look closer to the first term of above measure for a three-qubit state

$$\begin{aligned}
&\Lambda_{1,2}(\mathrm{M}_1)\overline{\Lambda}_{1,2}(\mathrm{M}_1) = \qquad (15) \\
&|\alpha_{1,1,1}\alpha_{2,1,2} - \alpha_{1,1,2}\alpha_{2,1,1}|^2 + |\alpha_{1,1,1}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,1}|^2 \\
&+|\alpha_{1,1,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,1}|^2 + |\alpha_{1,1,2}\alpha_{2,2,1} - \alpha_{1,2,1}\alpha_{2,1,2}|^2 \\
&+|\alpha_{1,1,2}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,1,2}|^2 + |\alpha_{1,2,1}\alpha_{2,2,2} - \alpha_{1,2,2}\alpha_{2,2,1}|^2.
\end{aligned}$$

$\Lambda_{1,2}(\mathrm{M}_2)\overline{\Lambda}_{1,2}(\mathrm{M}_2)$ and $\Lambda_{1,2}(\mathrm{M}_3)\overline{\Lambda}_{1,2}(\mathrm{M}_3)$ can be constructed in a similar ways. Finally, a measure of entanglement for three-qubit states is given by

$$\mathcal{E}(\mathcal{Q}_3^p(2,2,2)) = \left(\mathcal{N}_3 \sum_{j=1}^{3} \Lambda_{1,2}(\mathrm{M}_j)\overline{\Lambda}_{1,2}(\mathrm{M}_j)\right)^{1/2}. \quad (16)$$

This measure of entanglement for three-qubit states coincide with entanglement monotones given in Ref. [28].

5 Entanglement measure for general pure multipartite states

The generalization of our entanglement measure for multi-qubit states on general pure multipartite states can be done in a straightforward manner. Let an operator $\Lambda_{\mu,\nu}(\mathrm{M}_j) = \upsilon_\mu(\mathrm{M}_j) \wedge \upsilon_\nu(\mathrm{M}_j)$ be the wedge product between row number μ and ν of matrices M_j for all j, which is constructed by coefficient of a general quantum system $\mathcal{Q}_m^p(N_1,\ldots,N_m)$. For example, we have

$$\mathrm{M}_1 = \begin{pmatrix} \alpha_{1,1,\ldots,1} & \alpha_{1,1,\ldots,2} & \cdots & \alpha_{1,N_2,\ldots,N_m} \\ \alpha_{2,1,\ldots,1} & \alpha_{2,1,\ldots,2} & \cdots & \alpha_{2,N_2,\ldots,N_m} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{N_1,1,\ldots,1} & \alpha_{N_1,1,\ldots,N_m} & \cdots & \alpha_{N_1,N_2,\ldots,N_m} \end{pmatrix} \quad (17)$$

$\mathrm{M}_2\ldots,\mathrm{M}_m$ can be constructed in a similar ways by permutation of indices as in the case of multi-qubit states. Then we can define an entanglement measure for general pure multipartite states by

$$\mathcal{E}(\mathcal{Q}_m^p(N_1,\ldots,N_m)) = \left(\mathcal{N}_m \sum_{j=1}^{m} \sum_{\forall \nu>\mu=1} \Lambda_{\mu,\nu}(\mathrm{M}_j)\overline{\Lambda}_{\mu,\nu}(\mathrm{M}_j)\right)^{1/2}. \quad (18)$$

In this construction, the entanglement measure vanishes on product states and it is entanglement monotones. However this result need further investigation.

6 Conclusion

In this paper, we have derived a measure of entanglement that coincides with concurrence of a general pure bipartite state based on mapping of a tensor product space on an alternating tensor product space defined by a wedge product. Moreover, we have constructed a measure of entanglement for a pure multi-qubit state, which is entanglement monotones. Furthermore, we have generalized this construction into a general pure multipartite state.

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